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The Number of Analytic Solutions of a Singular Differential System*

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1. Consider the system

$$z^D y'(z) + B(z)\, y(z) = 0 \tag{1.1}$$

of differential equations in the complex plane. Here y is an n-dimensional vector, $B(z)$ is an $n \times n$ matrix of functions analytic at $z = 0$, and $D = \operatorname{diag}\{d_1, ..., d_n\}$, with $d_i = 0$, 1, or 2, $i = 1, ..., n$. A classical result for this system was given in 1926 by F. Lettenmeyer [6], who proved that (1.1) has at least $(n - \operatorname{tr} D)$ linearly independent solutions analytic at $z = 0$. For Lettenmeyer's theorem to be meaningful, of course we must have $\operatorname{tr} D < n$. Some results for the case in which $d_i = 2$, $i = 1, ..., n$, have been obtained by L. J. Grimm and L. M. Hall [1]. This result, like Lettenmeyer's, gives at best a lower bound for the number of analytic solutions.

In this paper we develop a procedure which yields the exact number of linearly independent solutions of (1.1) which are analytic at $z = 0$. This procedure will be applicable whenever D is as given above, i.e., D is an $n \times n$ diagonal matrix with some combination of zeros, ones and twos on the diagonal. Actually, no restriction need be placed on the size of the diagonal entries in D since, due to a result of H. L. Turrittin [7], the rank of a linear differential system can be reduced to rank one at the expense of increasing the dimension of the system. Rank one corresponds to $d_i = 2$, $i = 1, ..., n$. The techniques which we use are primarily based on the work of Grimm and Hall [2], the results of Hall [4], and the Cesari–Hale alternative problem technique (see Hale [3]).

2. Let E, $\bar{E}$, and C denote the open unit disc, closed unit disc, and unit circle, respectively. Let p be a nonnegative integer and define the Banach

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space A_p as the class of functions v analytic in E and p times continuously differentiable on $\bar{E}$ with norm given by

$$\| v \|_p = \{\max | v^{(i)}(z)| : i = 0,..., p,\ z \in C\}.$$

Let $A_{p,n}$ be the Banach space of n-vector functions $y(z) = (y^1(z), y^2(z),..., y^n(z))^T$ with $y^k(z) \in A_p$, $1 \leqslant k \leqslant n$, with norm

$$\| y \|_p^n = \{\max \| y^k \|_p ,\ 1 \leqslant k \leqslant n\}.$$

If f and g are n-vectors we will denote by (f, g) the "Euclidean inner product" of f and g, viz., $(f, g) = \sum_{i=1}^n f^i g^i$. (Here and in the rest of this paper superscripts will be used to denote the components of a vector, and subscripts will denote the coefficients in a power series.) The Hadamard product of f and g is then defined by

$$B(f, g; z) = \sum_{k=0}^{\infty} (f_k , g_k)\, z^k = \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} f_k^i g_k^i \right) z^k.$$

We can now define $A_{0,n}^*$, the space conjugate to $A_{0,n}$. For details see [2]. $A_{0,n}^*$ is the class of all n-vectors f, with components analytic in E, such that $\lim_{r\to 1^-} B(u, f; r)$ exists for each $u \in A_{0,n}$, with norm

$$\| f \|^* = \limsup_{r\to 1^-} \{| B(u, f; r)| : u \in A_{0,n} ,\ \| u \|_0^n = 1\}.$$

As indicated in [4], the Hardy space H^2 is closely related to the above spaces in the scalar case. In fact $A_0 \subset H^2 \subset A_0^*$ and, if f and g belong to H^2, $\langle f, \hat{g} \rangle = \lim_{r\to 1^-} B(f, g; r)$, where $\hat{g}(z) = \sum_{k=0}^{\infty} \bar{g}_k z^k$ and $\langle \cdot, \cdot \rangle$ denotes the H^2 inner product. Now let $H_n{}^2$ denote the space of n-vectors with components in H^2 and denote the H^2 norm by $\| \cdot \|$. Define a norm on $H_n{}^2$ by

$$\| f \|_n = \left(\sum_{i=1}^{n} \| f^i \|^2 \right)^{1/2}.$$

If $g(z) = \sum_{k=0}^{\infty} g_k z^k$ and $h(z) = \sum_{k=0}^{\infty} h_k z^k$ are n-vector functions analytic in E and if $\Phi(z) = \sum_{k=0}^{\infty} \Phi_k z^k$ is an $n \times n$ matrix function analytic in E define

$$[h, g] = \sum_{k=0}^{\infty} h_k{}^T g_k ,$$

and

$$[\Phi, g] = \sum_{k=0}^{\infty} \Phi_k g_k .$$

Note that $[h, g] = B(h, g; 1)$, but that $[\Phi, g]$ is an n-vector.

LEMMA 2.1. *Let $\Phi(z) = (\phi^{ij}(z))$, $i, j = 1, 2,\ldots, n$, be an $n \times n$ matrix with elements belonging to H^2, and let $g(z) \in H_n{}^2$. Then*

$$[\Phi, g] = \begin{bmatrix} \sum_{\ell=1}^{n} \langle \phi^{1\ell}, \hat{g}^{\ell} \rangle \\ \vdots \\ \sum_{\ell=1}^{n} \langle \phi^{n\ell}, \hat{g}^{\ell} \rangle \end{bmatrix}.$$

Proof. The ith row of $[\Phi, g]$ is

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{n} \phi_k^{i\ell} g_k{}^{\ell} = \sum_{\ell=1}^{n} \langle \phi^{i\ell}, \hat{g}^{\ell} \rangle$$

Finally, let the unit shift operator σ be defined by $\sigma F = zF$, where F is either a scalar, a vector, or a matrix.

3. Define the operator L: $A_{1,n} \to A_{0,n}$ by

$$Ly(z) = z^D y'(z) + B(z)\, y(z),$$

where D is as in Section 1. In [2] the following theorem is proved.

THEOREM 3.1. *The system $Ly = u$ has a solution in $A_{1,n}$ if and only if*

$$\lim_{r \to 1^-} B(u, f; r) = 0 \tag{3.1}$$

for all $f \in A_{0,n}^*$ such that

$$\lim_{r \to 1^-} B(Ly, f; r) = 0 \tag{3.2}$$

for all $y \in A_{1,n}$.

Since f is analytic in E and each element of $B(z)$ is in A_0 we can write $f(z) = \sum_{k=0}^{\infty} f_k z^k$, $B(z) = \sum_{k=0}^{\infty} B_k z^k$, and note that (3.2) is equivalent to the infinite system

$$\begin{aligned} &\sum_{k=0}^{\infty} B_k f_k = 0 \\ &\sum_{k=0}^{\infty} B_k f_{k+1} = -(f_{d_1}^1, f_{d_2}^2, \ldots, f_{d_n}^n)^T \\ &\quad\vdots \\ &\sum_{k=0}^{\infty} B_k f_{k+m} = -m(f_{d_1+m-1}^1, f_{d_2+m-1}^2, \ldots, f_{d_n+m-1}^n)^T \\ &\quad\vdots \end{aligned} \tag{3.3}$$

Note that (3.2) or (3.3) defines the cokernel of L.

We now proceed to express (3.3) in terms of operators. Two different types of operators will be used, "ordinary" $n \times n$ matrices, and operators given in terms of infinite matrices. In the latter case we will "stack" the coefficients of the vector function which is being operated on to obtain a sequence. We shall define the following operators.

$$
\begin{aligned}
I_n &= n \times n \text{ identity},\\
I_\infty &= \operatorname{diag}\{1, 1, 1, \ldots\},\\
E_j &= n \times n \text{ matrix which is the projection onto the}\\
&\qquad \text{null space of } (D - jI_n), \qquad j = 0, 1, 2, \ldots .\\
U_0 &= \operatorname{subdiag}\{0, -I_n, -2I_n, -3I_n, \ldots\},\\
U_1 &= \operatorname{diag}\{0, -I_n, -2I_n, -3I_n, \ldots\},\\
U_2 &= \operatorname{superdiag}\{0, -I_n, -2I_n, -3I_n, \ldots\},\\
U &= E_0 U_0 + E_1 U_1 + E_2 U_2\\
W &= \begin{bmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ 0 & B_0 & B_1 & B_2 & \cdots \\ 0 & 0 & B_0 & B_1 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}\\
P &= \operatorname{diag}\{I_n, E_2 I_n, 0, 0, \ldots\},\\
Q &= \operatorname{diag}\{0, (E_1 + E_2) I_n, I_n, \ldots\},\\
M_0 &= \operatorname{superdiag}\{0, -I_n, -\tfrac{1}{2}I_n, -\tfrac{1}{3}I_n, \ldots\},\\
M_1 &= \operatorname{diag}\{0, -I_n, -\tfrac{1}{2}I_n, -\tfrac{1}{3}I_n, \ldots\},\\
M_2 &= \operatorname{subdiag}\{0, -I_n, -\tfrac{1}{2}I_n, -\tfrac{1}{3}I_n, \ldots\},\\
M &= E_0 M_0 + E_1 M_1 + E_2 M_2\\
P_N &= \operatorname{diag}\{I_n, I_n, \ldots, I_n, 0, \ldots\}, (N + 1)\, I_n\text{'s}.
\end{aligned}
$$

Note that P is the projection onto the kernel of U and Q is the projection onto the range of U. Also, $UM = Q$ and $MU = I_\infty - P$. Now let $P_N f = g$ and $(I_\infty - P_N) f = h$. System (3.3) can now be written as

$$
\begin{aligned}
&\text{(a)} \quad h = (I_\infty - P_N)\, MQW(g + h),\\
&\text{(b)} \quad (I_\infty - P) g = P_N MQW(g + h), \qquad\qquad (3.4)\\
&\text{(c)} \quad (I_\infty - Q)\, W(g + h) = 0.
\end{aligned}
$$

This decomposition is similar to that done in [4] for the scalar case. We shall also need the representation of the matrix MQW, which is given as follows:

$MQW =$

$$
-\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & E_1B_0 & (E_0B_0+E_1B_1) & (E_0B_1+E_1B_2) & (E_0B_2+E_1B_3) & \cdots \\
0 & E_2B_0 & (\frac{1}{2}E_1B_0+E_2B_1) & (\frac{1}{2}E_0B_0+\frac{1}{2}E_1B_1+E_2B_2) & (\frac{1}{2}E_0B_1+\frac{1}{2}E_1B_2+E_2B_3) & \cdots \\
0 & 0 & \frac{1}{2}E_2B_0 & (\frac{1}{3}E_1B_0+\frac{1}{2}E_2B_1) & (\frac{1}{3}E_0B_0+\frac{1}{3}E_1B_1+\frac{1}{2}E_2B_2) & \cdots \\
0 & 0 & 0 & \frac{1}{3}E_2B_0 & (\frac{1}{4}E_1B_0+\frac{1}{3}E_2B_1) & \cdots \\
0 & 0 & 0 & 0 & \frac{1}{4}E_2B_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

THEOREM 3.2. *For sufficiently large N the operator $(I_\infty - P_N)MQW$ is a contraction on H_n^2.*

Proof. For $f \in H_n^2$ we have

$$
(I_\infty - P_N)MQWf = -\sum_{k=N+1}^{\infty} \left\{\left(\frac{1}{k-1}\right)[\sigma^{k-1}E_2B, f] + \left(\frac{1}{k}\right)[\sigma^k E_1B, f] + \left(\frac{1}{k}\right)[\sigma^{k+1}E_0B, f]\right\} z^k.
$$

So by Lemma 2.1 we get

$$
\begin{aligned}
&\|(I_\infty - P_N)MQWf\|_n^2 \\
&\leqslant \sum_{k=N+1}^{\infty} \left\{\left(\frac{1}{k-1}\right)^2 \left|\sum_{i=1}^{n}\sum_{\ell=1}^{n} \langle \sigma^{k-1}(E_2B)^{i\ell}, \hat{f}^\ell\rangle\right|^2 \right. \\
&\quad + \left(\frac{1}{k}\right)^2 \left|\sum_{i=1}^{n}\sum_{\ell=1}^{n} \langle \sigma^{k}(E_1B)^{i\ell}, \hat{f}^\ell\rangle\right|^2 \\
&\quad \left. + \left(\frac{1}{k}\right)^2 \left|\sum_{i=1}^{n}\sum_{\ell=1}^{n} \langle \sigma^{k+1}(E_0B)^{i\ell}, \hat{f}^\ell\rangle\right|^2 \right\} \\
&\leqslant \left(\sum_{k=N}^{\infty} \frac{1}{k^2}\right) \sum_{i=1}^{n}\sum_{\ell=1}^{n} \{\|(E_2B)^{i\ell}\|^2 + \|(E_1B)^{i\ell}\|^2 + \|(E_0B)^{i\ell}\|^2\} \|f^\ell\|^2 \\
&\leqslant \left(\sum_{k=N}^{\infty} \frac{1}{k^2}\right) 3\,\|B\|^2 \|f\|_n^2 .
\end{aligned}
$$

Here we have denoted the element in the ith row and ℓth column of E_jB, $j = 0, 1, 2$, by $(E_jB)^{il}$. Also, if $B(z) = ((B)^{il}(z))$ is an $n \times n$ matrix with elements in H_n^2 we define $\|B\| = (\sum_{i,l-1}^{n} \|(B)^{il}\|^2)^{1/2}$. The proof is now complete.

As a result of Theorem 3.2 we have

$$[I_\infty - (I_\infty - P_N) MQW]^{-1} = \sum_{k=0}^{\infty} [(I_\infty - P_N) MQW]^k,$$

and so it is now possible to solve (3.4a) for h in terms of g:

$$h = [I_\infty - (I_\infty - P_N) MQW]^{-1}(I_\infty - P_N) MQWg. \tag{3.5}$$

We now substitute this h into (3.4b) to obtain

$$(I_\infty - P)g = \left\{P_N MQW \sum_{k=0}^{\infty} [(I - P_N) MQW]^k\right\} g,$$

which can be expressed as the following system of linear equations in the n-vector unknowns $g_1 ,..., g_N$:

$$\begin{gathered} (E_0 + E_1) g_1 + [\sigma^2 E_0 B, g] + [\sigma E_1 B, g] + C_1 g_N = 0 \\ g_2 + \tfrac{1}{2}[\sigma^3 E_0 B, g] + \tfrac{1}{2}[\sigma^2 E_1 B, g] + [\sigma E_2 B, g] + C_2 g_N = 0 \\ \vdots \\ g_N + \frac{1}{N} [\sigma^N E_1 B, g] + \frac{1}{N-1} [\sigma^{N-1} E_2 B, g] + C_N g_N = 0, \end{gathered} \tag{3.6}$$

where $C_1 = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} (E_0 B_{N+j-1} + E_1 B_{N+j}) \cdot (N + j + 2, N + 1)^k$, and $C_i = \sum_{i=1}^{\infty} \sum_{j=0}^{k-1} (E_0 B_{N+j-i} + E_1 B_{N+j-i+1} + E_2 B_{N+j-i+2}) \cdot (N + j + 2, N + 1)^k$, $2 \leqslant i \leqslant N$ and $(N + i, N + 1)^k$ denotes the $n \times n$ matrix in the $(N + i, N + 1)$ position of $[(I_\infty - P_N) MQW]^k$, where we consider an element of $[(I_\infty - P_N) MQW]^k$ to be an $n \times n$ matrix. We shall now consider (3.6) as a system of nN scalar equations in the nN unknowns $g_k^j, j = 1,..., n, k = 1,..., N$, and write the system as

$$\mathscr{B}\tilde{g} = 0, \tag{3.7}$$

where $\tilde{g} = (g_1^1, g_1^2,..., g_1^n, g_2^1,..., g_N^n)^T$ and $\mathscr{B}$ is the $nN \times nN$ matrix obtained from (3.6). If rank $\mathscr{B} = r$ then we can solve for r of the $g_k^j s$ in terms of the remaining $(nN - r)$ $g_k^j s$. Let $\rho = nN - r$, and let G be a ρ-vector whose components are the arbitrary $g_k^j s$ (in an arbitrary but fixed order). Then we can express $g(z)$, except for g_0, in terms of G as follows:

$$g(z) = g_0 + \sum_{k=1}^{N} M_k G z^k.$$

Here, for each k, M_k is the appropriate $n \times \rho$ matrix determined by our solution of (3.7). Careful scrutiny of the matrices involved in (3.5) reveals that the coefficients of h depend only on g_N, and so we can write, noting that $f_0 = g_0$,

$$f(z) = f_0 + \sum_{k=1}^{\infty} M_k G z^k.$$

We now turn to (3.4c) which can be expressed as the following system:

$$\begin{bmatrix} B_0 & \sum_{i=1}^{\infty} B_i M_i \\ 0 & \sum_{i=1}^{\infty} E_0 B_{i-1} M_i \end{bmatrix} \begin{bmatrix} f_0 \\ G \end{bmatrix} = 0. \tag{3.8}$$

Denote the $2n \times (\rho + n)$ matrix in (3.8) by $\mathscr{C}$ and let $\gamma = \operatorname{rank} \mathscr{C}$. Hence we can solve (3.8) for γ unknowns in terms of either the remaining $n + \rho - \gamma$ unknowns if $\rho \leqslant n$, or the remaining $2n - \gamma$ unknowns if $\rho \geqslant n$.

Some special cases now need to be examined.

Case 1. $\rho = 0$. This requires $r = nN$ which means $E_2 \equiv 0$, because otherwise one or more of the first n rows of $\mathscr{B}$ would be all zeros. $E_2 \equiv 0$ is equivalent to D containing only 0's and 1's on the diagonal and this will result in the $(N + 1)$st block column of $(I_\infty - P_N)\, MQW$ being all zeros which implies that $C_i = 0$, $i = 1,\ldots, N$. Hence system (3.6) can be solved from the bottom up and we get, in addition to the fact that $g_N = g_{N-1} = \cdots = g_1 = 0$, that the matrices $[kI_n + E_1 B_0]$, $k = 2,\ldots, N$, and the matrix $[E_0 + E_1(I_n + B_0)]$ are all nonsingular. System (3.8) now becomes $B_0 f_0 = 0$ and so $\gamma = \operatorname{rank} B_0$.

Case 2. $\rho = n$. In this case the matrix $\mathscr{C}$ is a $2n \times 2n$ matrix and G is an n-vector. If $\gamma = 2n$ then (3.8) has only the trivial solution, which means (3.4) has only the trivial solution, as will be shown later.

Case 3. $0 < \rho < n$. We first note that this cannot occur when $D = 2I$, for then $r \leqslant n(N - 1)$ and $\rho \geqslant n$. When $0 < \rho < n$ system (3.8) may represent a system with more equations than unknowns and thus could have only the trivial solution. We shall show that if (3.8) has only the trivial solution then B_0 is nonsingular, but that the converse is false.

If (3.8) has only the trivial solution then rank $\mathscr{C}$ must be $n + \rho$. Otherwise, the columns of $\mathscr{C}$ would be linearly dependent and there would exist a nontrivial linear combination of the columns equal to zero. The coefficients of this linear combination would give a nonzero vector v such that $\mathscr{C}v = 0$, a contradiction. If the columns of $\mathscr{C}$ are linearly independent then so are the columns of B_0, so B_0 is nonsingular.

Also, it is clear that if $\operatorname{rank} \mathscr{C} = n + \rho$ then (3.8) has only the trivial solution. However, B_0 nonsingular does not guarantee that $\operatorname{rank} \mathscr{C} = n + \rho$. Finally, we note that if $E_0 \equiv I_n$ then $\rho = 0$, which was covered in Case 1, and if $E_0 \equiv 0$ then $\mathscr{C}$ is essentially an $n \times (n + \rho)$ dimensional matrix and the possibility of more equations than unknowns is eliminated.

We have thus found $n + \rho - \gamma$ linearly independent solutions of (3.2) which belong to $H_n{}^2$, and shall next show that our procedure yields all solutions of (3.2). Let f^* be a solution of (3.2) with $g^* = P_N f^*$ and $h^* = (I_\infty - P_N) f^*$. Since f^* satisfies (3.4a) we can write

$$[I_\infty - (I_\infty - P_N) MQW] h^* = (I_\infty - P_N) MQWg^*. \tag{3.10}$$

The right hand side of (3.10) is easily calculated to be $-(1/N) E_2 B_0 g_N^* z^{N+1}$, which is in H_n^2, and so we can operate on both sides of (3.10) with $[I_\infty - (I_\infty - P_N) MQW]^{-1}$. Hence $h^* \in H_n^2$ and so $f^* \in H_n^2$ also. We are now ready to state our main result which summarizes the above work.

THEOREM 3.3. *Let L, ρ, and γ be as defined in this section, and let $d = \operatorname{tr} D$. Then*

(i) $\operatorname{coker} L$ *is a subset of* H_n^2,

(ii) $\dim \operatorname{coker} L = n + \rho - \gamma$,

(iii) $\dim \ker L = 2n + \rho - \gamma - d = n - d + \dim \operatorname{coker} L$.

Conclusion (iii) of Theorem 3.3 is obtained from the fact that $\dim \operatorname{coker} L - \dim \ker L = \operatorname{index} L = d - n$ (see [2] for details). The following corollary gives a necessary and sufficient condition for $\dim \ker L$ to equal the lower bound given by Lettenmeyer's theorem. This corollary was proved by Grimm and Hall for a slightly more general operator in [2], but now if there are more solutions than the number guaranteed by Lettenmeyer, Theorem 3.3 tells us exactly how many more.

COROLLARY 3.3.1. *Given the hypotheses of Theorem* 3.3, *and* $d < n$, $\dim \ker L = n - d$ *if and only if* $\dim \operatorname{coker} L = 0$.

The following corollaries give some results based on the structure of the matrix D.

COROLLARY 3.3.2. *If* $E_2 \equiv 0$ *and f belongs to* $\operatorname{coker} L$ *then f is a polynomial which satisfies* (3.7) *and* (3.8).

As a result of Corollary 3.3.2 we now know that the polynomials constructed in the proof of Theorem 4.3, [5], span $\operatorname{coker} L$ in the case $D = I_n$. The next corollary also relates to some of the results in [5].

COROLLARY 3.3.3. *Let* $D = I_n$. *Then* $\dim \ker L = \dim \operatorname{coker} L$ *and*

$$\dim \ker L = \delta, \qquad \text{where} \quad \delta = n(N+1) - \operatorname{rank} \mathscr{B}_1 \quad \text{and}$$

$$\mathscr{B}_1 = \begin{bmatrix} B_0 & B_1 & \cdots & B_n \\ 0 & (B_0 + I_n) & \cdots & B_{N-1} \\ \vdots & & \ddots & \\ \vdots & & & \\ 0 & \cdots & & (B_0 + NI_n) \end{bmatrix}$$

COROLLARY 3.3.4. *Let $D = 2I_n$. Then the following are equivalent*:

(i) $\rho = \gamma = n$,

(ii) $\dim \operatorname{coker} L = n$,

(iii) $\dim \ker L = 0$.

4. In this section we shall give some examples to illustrate the above results.

EXAMPLE 1. Bessel's equation of order ν, with ν a nonnegative integer. As a two-dimensional system, Bessel's equation is

$$z^{I_2} y'(z) + B(z)\, y(z) = 0 \tag{4.1}$$

where $B(z) = \begin{pmatrix} 0 & -1 \\ -\nu^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z^2$. It is easy to see that, in Corollary 3.3.3, $\delta = 1$ when $0 \leqslant \nu \leqslant N$. In fact, since Bessel's equation of integer order has exactly one analytic solution at $z = 0$, we will always have $N \geqslant \nu$. Note that Lettenmeyer's theorem is not applicable, for practical purposes, to this example because $\operatorname{tr} D = n = 2$. Also, $\operatorname{coker} L$, L as in (4.1), consists only of polynomials by Corollary 3.3.2, and this polynomial (there is only one because $\dim \operatorname{coker} L = 1$) is constructed in [5].

EXAMPLE 2. Let $n = 3$, $D = 2I_3$, and

$$B(z) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - I_3 z.$$

We have

$$\mathscr{B} = \left[\begin{array}{ccccccccccccc}
0 & 0 & 1 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
 & & & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & & & & \\
 & & & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & & & & \\
 & & & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & & & & \\
 & & & & & & & \ddots & & \ddots & & & \\
 & & & & & & & 0 & 0 & \frac{1}{N-1} & \frac{N-2}{N-1} & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & \frac{N-2}{N-1} & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & \frac{N-2}{N-1}
\end{array}\right]$$

and $\rho = 5$ since $r = 3N - 5$. We will calculate $g(z)$ for this example. System (3.7) can be solved in terms of g_1^1, g_1^2, g_2^1, g_2^2, and g_2^3 to get

$$\tilde{g}(z) = \begin{pmatrix} g_1^1 \\ g_2^1 \\ 0 \end{pmatrix} z + \begin{pmatrix} g_2^1 \\ g_2^2 \\ g_2^3 \end{pmatrix} z^2 + \begin{pmatrix} -g_2^3 \\ 0 \\ 0 \end{pmatrix} z^3.$$

The matrix $\mathscr{C}$ is now found to be

$$\mathscr{C} = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and so $\gamma = 2$, and $g(z)$ is given by

$$g(z) = \begin{pmatrix} g_0^1 \\ g_0^2 \\ -g_1^1 \end{pmatrix} + \begin{pmatrix} g_1^1 \\ 0 \\ 0 \end{pmatrix} z + \begin{pmatrix} g_2^1 \\ g_2^2 \\ g_2^3 \end{pmatrix} z^2 + \begin{pmatrix} -g_2^3 \\ 0 \\ 0 \end{pmatrix} z^3.$$

Using (i) and (ii) in Theorem 3.3 we see that $\dim \operatorname{coker} L = 3 + 5 - 2 = 6$, and $\dim \ker L = 6 + 5 - 2 - 6 = 3$ for this example. The fact that $\dim \ker L = 3$ and $n = 3$ means that here $z = 0$ is an apparent singular point.

Example 3. Let $D = \operatorname{diag}\{0, 1, 1\}$, and $B(z) = \operatorname{diag}\{-1, -1, 1\}$. Here $E_0 = I_3 - D$ and $E_1 = D$. As noted earlier, since $E_2 \equiv 0$ we get $C_i \equiv 0$, $i = 1,\ldots, N$, in (3.6). The matrix $\mathscr{B}$, obtained from (3.6), is

$$\mathscr{B} = \operatorname{diag}\left\{1, 0, 2, 1, \frac{1}{2}, \frac{3}{2}, 1, \frac{2}{3}, \frac{4}{3}, \ldots, 1, \frac{N-1}{N}, \frac{N+1}{N}\right\}$$
$$+ \operatorname{super}^3 \operatorname{diag}\left\{-1, 0, 0, -\frac{1}{2}, 0, 0, \ldots, -\frac{1}{N-1}, 0, 0\right\},$$

where $\operatorname{super}^3 \operatorname{diag}\{\cdots\}$ indicates a matrix with zeros everywhere except on the third diagonal above the main diagonal. For this $\mathscr{B}$, $\operatorname{rank} \mathscr{B} = 3N - 1$, and so $\rho = 1$. We solve (3.6) in terms of g_1^2 and obtain $g(z) = g_0 + (0, g_1^2, 0)^T z$, so that $G = g_1^2$, $M_1 = (0, 1, 0)^T$, and $M_i = 0$, $i \geqslant 2$. Hence the matrix $\mathscr{C}$ is given by

$$\mathscr{C} = \begin{pmatrix} B_0 & 0 \\ 0 & E_0 B_0 M_1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and we see that $\gamma = 3$. This means that

$$g(z) = M_1 Gz. \tag{4.2}$$

Applying Theorem 3.3 we get that $\dim \operatorname{coker} L = 1$ and $\dim \ker L = 2$ for this example. Note that by Corollary 3.3.2 the cokernel of L is completely characterized by the g in (4.2). Also note that Lettenmeyer's theorem guarantees the existence of just one analytic solution for this example.

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